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LAGRANGE'S SEXTIC.

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It has been for a long time considered that Lagrange's method for the reduction of the quintic equation depended upon the solution of an auxiliary equation of the sixth degree, known and designated by him as a "Sextic Resolvent." While this equation has been the subject of a good deal of valuable literature, and, with its analogues, occupies an important place in the theory of equations, I am not aware that it has been shown that this resolvent of a higher degree is not necessary to the application of Lagrange's method. It is the object of the present paper to show that Lagrange's "Sextic Resolvent" is not necessary to the resolution of the quintic equation in the application of his own method for its reduction, and neither does it require a resolvent equation of a higher degree than the fifth degree for this purpose.

Denoting the five roots of the quintic by a, b, c, d, e, and the roots of $\omega^5 - 1 = 0$ by 1, ω, ω^2, ω^3, ω^4, we may write the entire six Lagrangian groups as follows:

$$M_1 \quad \begin{array}{l} (a + \omega b + \omega^2 c + \omega^3 d + \omega^4 e)^5 \\ (a + \omega^2 b + \omega^4 c + \omega d + \omega^3 e)^5 \\ (a + \omega^3 b + \omega c + \omega^4 d + \omega^2 e)^5 \\ (a + \omega^4 b + \omega^3 c + \omega^2 d + \omega e)^5 \end{array}$$

$$N_1 \quad \begin{array}{l} (a + \omega c + \omega^2 b + \omega^3 d + \omega^4 e)^5 \\ (a + \omega^2 c + \omega^4 b + \omega d + \omega^3 e)^5 \\ (a + \omega^3 c + \omega b + \omega^4 d + \omega^2 e)^5 \\ (a + \omega^4 c + \omega^3 b + \omega^2 d + \omega e)^5 \end{array}$$

$$P_1\quad\begin{array}{l}(a+\omega d+\omega^2c+\omega^3b+\omega^4e)^5\\(a+\omega^2d+\omega^4c+\omega b+\omega^3e)^5\\(a+\omega^3d+\omega c+\omega^4b+\omega^2e)^5\\(a+\omega^4d+\omega^3c+\omega^2b+\omega e)^5\end{array}$$

$$Q_1\quad\begin{array}{l}(a+\omega e+\omega^2c+\omega^3d+\omega^4b)^5\\(a+\omega^2e+\omega^4c+\omega d+\omega^3b)^5\\(a+\omega^3e+\omega c+\omega^4d+\omega^2b)^5\\(a+\omega^4e+\omega^3c+\omega^2d+\omega b)^5\end{array}$$

$$R_1\quad\begin{array}{l}(a+\omega b+\omega^2e+\omega^3d+\omega^4c)^5\\(a+\omega^2b+\omega^4e+\omega d+\omega^3c)^5\\(a+\omega^3b+\omega e+\omega^4d+\omega^2c)^5\\(a+\omega^4b+\omega^3e+\omega^2d+\omega c)^5\end{array}$$

$$S_1\quad\begin{array}{l}(a+\omega b+\omega^2c+\omega^3e+\omega^4d)^5\\(a+\omega^2b+\omega^4c+\omega e+\omega^3d)^5\\(a+\omega^3b+\omega c+\omega^4e+\omega^2d)^5\\(a+\omega^4b+\omega^3c+\omega^2e+\omega d)^5\end{array}$$

Any one of the 24 values here written is a root of a quartic consequent upon the solution of a sextic; and if we consider the four imaginary values corresponding to the roots of unity, viz. ω, ω^2, ω^3, ω^4, as the four values of group M_1, any symmetrical function of these, or rather their fifth powers, as for instance their sum, is a six-valued function of the roots, and is therefore determined by a sextic equation.

If we separate these six groups into three pairs, viz.:

$$w_1 = M_1 + Q_1, \qquad w_2 = N_1 + S_1, \qquad w_3 = P_1 + R_1,$$

these pairs do not break up for any of the substitutions of the roots b, c, d, e; the pairs simply interchange places. For example, substitute in these six groups the roots b and c and we have, M_1 and N_1 change places, Q_1 and S_1 change places, P_1 and R_1 change places, or w_1 and w_2 change places and w_3 remains constant.

The substitution of the groups caused by all the substitutions of the roots *b*, *c*, *d*, *e* may be seen in the following table:

	bc	*bd*	*be*	*cd*	*ce*	*de*
$M_1 + Q_1$	$N_1 + S_1$	$P_1 + R_1$	$Q_1 + M_1$	$Q_1 + M_1$	$R_1 + P_1$	$S_1 + N_1$
$N_1 + S_1$	$M_1 + Q_1$	$S_1 + N_1$	$P_1 + R_1$	$P_1 + R_1$	$S_1 + N_1$	$Q_1 + M_1$
$P_1 + R_1$	$R_1 + P_1$	$M_1 + Q_1$	$N_1 + S_1$	$N_1 + S_1$	$Q_1 + M_1$	$R_1 + P_1$

from which it is evident that the pairs in all the columns remain unchanged for these substitutions; hence any symmetrical functions of these pairs or any symmetrical function of w_1, w_2, w_3 is a symmetrical function of *b*, *c*, *d*, *e*, and peradventure said function should also be a symmetrical function of M_1, N_1, P_1, Q_1, R_1, S_1, it would also be a symmetrical function of all five of the roots *a*, *b*, *c*, *d*, *e*.

Now, if w_1, w_2, w_3 be the roots of

$$W^3 - A_1 W^2 + B_1 W - C_1 = 0,$$

A_1 is evidently a symmetrical function of the roots of the quintic, and B_1 and C_1 are symmetrical functions of *b*, *c*, *d*, and *e* only, and therefore change for the substitutions *ab*, *ac*, *ad*, and *ae*, and are therefore five-valued functions of the roots, whose determination therefore requires the solution of a quintic equation, whence the roots w_1, w_2, w_3 become known.

Next let the quartic roots in group M_1 be taken as the sum of their products two and two at a time, and so on for all the other groups, thus making six other groups M_2, N_2, P_2, Q_2, R_2, S_2, the subscripts denoting the number of roots taken together in each product.

Denoting as before the similar pairs

$$x_1 = M_2 + Q_2, \qquad x_2 = N_2 + S_2, \qquad x_3 = P_2 + R_2,$$

we have $X^3 - A_2 X^2 + B_2 X - C_2 = 0$, whence by the aid of subordinate quintics as before the values x_1, x_2, x_3 become known. Similarly, treating the groups in sums of the products of the quartic roots taken three at a time and four at a time, we have

$y_1 = M_3 + Q_3$, $y_2 = N_3 + S_3$, $y_3 = P_3 + R_3$, whence y_1, y_2, y_3 become known;
$z_1 = M_4 + Q_4$, $z_2 = N_4 + S_4$, $z_3 = P_4 + R_4$, " z_1, z_2, z_3 " "

Instead of taking the above pairs as their sums, it is obvious that the above reasoning may be applied to their products as well; thus, letting

$$\begin{array}{lll} w'_1 = M_1Q_1, & w'_2 = N_1S_1, & w'_3 = P_1R_1; \\ x'_1 = M_2Q_2, & x'_2 = N_2S_2, & x'_3 = P_2R_2; \\ y'_1 = M_3Q_3, & y'_2 = N_3S_3, & y'_3 = P_3R_3; \\ z'_1 = M_4Q_4, & z'_2 = N_4S_4, & z'_3 = P_4R_4; \end{array}$$

we should have by the aid of auxiliary equations of the fifth degree, as above, the quantities w'_1, w'_2, w'_3; x'_1, x'_2, x'_3; y'_1, y'_2, y'_3; z'_1, z'_2, z'_3 all ultimately determined in terms of the coefficients of the original quintic equation.

Evidently by the solution of

$$\begin{array}{ll} \theta^2 - w_1\theta + w'_1 = 0, & M_1 \text{ becomes known;} \\ \theta^2 - x_1\theta + x'_1 = 0, & M_2 \quad \text{``} \qquad \text{``} \\ \theta^2 - y_1\theta + y'_1 = 0, & M_3 \quad \text{``} \qquad \text{``} \\ \theta^2 - z_1\theta + z'_1 = 0, & M_4 \quad \text{``} \qquad \text{``} \end{array}$$

whence the four quartic roots of group M_1 are given by the quartic equation

$$\psi^4 - M_1\psi^3 + M_2\psi^2 - M_3\psi + M_4 = 0.$$

Thus without the aid of any intervening equations higher than the fifth degree we arrive at the final quartic equation, whose roots are evidently identically the *same* as the roots of the quartic employed by Lagrange after the resolution of his "Sextic."